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MATHEMATICAL MODELING OF TRAJECTORIES CONSTRUCTION, MOVEMENT OF THE GRIPPING DEVICE OF A COLLABORATIVE ROBOT

Abstract. The object of the study is the process of constructing and analyzing the trajectories of the gripping device of a collaborative robot-manipulator under spatial constraints and the presence of obstacles in a dynamic environment. The subject of the study is mathematical models, algorithmic and software for modeling the optimal motion of the manipulator end effector taking into account kinematic, dynamic and energy constraints. The aim of the research is to construct trajectories of the collaborative robot's gripping device, taking into account constraints and optimal control actions in continuous time, which ensure the construction of trajectories with minimal energy consumption, compliance with given spatial constraints, and avoidance of collisions with obstacles. The research methodology is based on the application of the Pontryagin maximum principle to form the conditions for optimal control and the construction of a system of differential equations with boundary conditions. A special cost functional has been developed to quantify energy consumption and take into account penalties for approaching prohibited zones. The numerical solution of the problem was implemented using the Euler method, and the optimization of the trajectory parameters with fixed final effector coordinates was implemented using the least squares method with constraints. The Python programming language and the Matplotlib library were used to visualize the results. As a result of the study, optimal trajectories of the gripping device were obtained, which ensure collision avoidance, compliance with spatial constraints, and reduced energy consumption when reaching the specified final effector positions. The simulation confirmed the effectiveness of the developed method and its resistance to changes in environmental parameters. The conclusions of the study indicate that the proposed approach allows for a comprehensive solution to the problem of planning the movement of collaborative robots in the optimal control mode taking into account constraints. The results obtained can be applied in Industry 5.0 production systems, robotic service complexes, automated warehouse systems, and robots that interact with humans in a limited space.

Keywords: collaborative robot; robot manipulator; trajectory of the roc; gripping device; mathematical modeling; Pontryagin's maximum principle.

Introduction

Formulation of the problem. In today's rapidly evolving technology landscape and the transition to a new manufacturing paradigm, defined as Industry 5.0, approaches to automation, human-machine interaction, and robotic system control are undergoing a transformation [1–3]. The main idea of Industry 5.0 is to harmoniously combine human creativity and adaptability with the precision and power of robotic technologies, which involves the development of flexible, safe, and intelligent control systems [4–6]. In this context, the study of mathematical methods for describing the trajectories of collaborative manipulator robots, which work in a common space with a person and must ensure high accuracy, efficiency and safety when performing complex tasks, is of particular relevance [7–9]. In the context of flexible production focused on individualized products, there is a need for algorithms that can take into account physical limitations, design features and dynamics of manipulator movement, as well as optimize control actions in real time [10, 11]. The formation of mathematical models that allow describing the continuous motion of the gripping device taking into account energy, kinematic and dynamic constraints opens up opportunities for building effective control systems that are able to adapt to changes in the environment and the features of interaction with a person [12, 13]. The application of the Pontryagin maximum principle to build such models allows formalizing the optimal control problem in the form of a system of differential equations that simultaneously take into account the objective functions and system constraints [14].

The relevance of this approach is due to the need to create mathematically based real-time solutions that can be integrated into the software of collaborative robots and ensure uninterrupted operation in a dynamic production environment. Therefore, the study of the mathematical description of constructing the trajectories of the collaborative robot's gripping device, taking into account constraints and optimal control actions in continuous time, is extremely important in the context of implementing the concepts of Industry 5.0 and ensuring a new quality of interaction between humans and robotic systems in cyber-physical production [15–17].

Analysis of recent research and publications. In the work of Elumalai, V. K., a deep reinforcement learning based on the Proximal Policy Optimization algorithm is proposed for the problem of trajectory tracking of a flexible robotic manipulator, which allows to implement an optimal control policy in a changing environment [18]. This solution makes it possible to create dynamically adaptive control strategies, which can be partially used in constructing optimal control of the gripping device, although there is no specific attention to continuous-time constraints. In the article of Lai, J., the kinematic features of flexible robots for minimally invasive surgery are investigated, with an emphasis on positioning accuracy and complex geometry of the manipulators [19].

This study can be useful from the point of view of geometric modeling and motion constraints, but does not cover the aspect of optimal control or mathematical description of dynamics. In the paper Hu, S. developed an adaptive sliding control method with model compensation for accurate trajectory tracking, which works in continuous time and takes into account model errors [20]. The proposed approach can be integrated into the dynamic model of the gripper as a lower control level, since it provides high accuracy and robustness to uncertainties. In the paper Shen, N. presented a method for incremental multi-criteria optimization of the trajectory in real time for collaborative robots, which allows taking into account the goals of safety, productivity and smoothness [21]. This solution can be directly used for constructing trajectories taking into account constraints and is relevant to the topic of the article.

This solution can be directly used to construct trajectories taking into account constraints and is relevant to the topic of the article. In the work of Shrivastava, A., a generalized review of approaches to manipulator motion planning is carried out, including the analysis of interpolation methods, velocity constraints, and acceleration, which provides a broad basis for choosing effective strategies [11]. The results obtained can be used to formalize the requirements for trajectories and choose optimality metrics. In the study of Ji, Y., a trajectory planning algorithm for moving manipulators in space using an improved GBNN algorithm that takes into account spatial constraints and configuration changes [22]. The technique can be applied to modeling complex workspaces, but requires adaptation for continuous-time tasks. In the article Wang, Y. proposed a control method using adaptive dynamic programming and multi-criteria optimal feedback for manipulators with variable configuration [23]. This technique allows to take into account both force and positional constraints and is valuable for building optimal control of the gripping device. In the article Zhang, H. investigated multi-agent cooperation between robots to estimate the mass of the load, which allows to improve the control accuracy during joint manipulation of objects [24]. Although this model is focused on teamwork, the principles of collective tracking can be integrated into systems for assessing external influences on the gripping device. In the article Huang, S. presented a new approach to simulation learning, in which a person purposefully creates perturbations to improve robot learning [25]. Although the work is focused on learning behavioral models, the mechanism for taking into account human actions can be valuable in creating adaptive trajectory planning in conditions of interaction with the operator. In the work of Sun, E., a system of macro-mini manipulators for welding in narrow environments is considered, which requires high accuracy of motion planning under physical constraints [26]. This solution confirms the relevance of trajectory planning taking into account constraints and can be used as an example of effective distribution of movements between links. In the article of Becker, M., a reactive model of global obstacle avoidance in the form of informed circular fields is proposed, which allows the manipulator to adaptively adjust the trajectory in real time [27]. This solution should be integrated into the optimal trajectory correction system when external threats or constraints arise.

Summarizing the analysis, it can be stated that there is significant scientific and practical interest in the problems of trajectory planning and optimal control of manipulators, especially under continuous-time conditions and environmental constraints. However, most of the research is either focused on specific types of manipulators (flexible, minimally invasive), or focuses on planning algorithms without full integration of dynamics, constraints, and real-time. This emphasizes the relevance of the topic of building a generalized mathematical model that takes into account kinematic constraints, motion dynamics, optimal control actions, and features of the gripping device in continuous-time conditions to improve the safety, efficiency, and accuracy of the collaborative robot-manipulator.

Formulation of the purpose of the article (statement of the task). The purpose of the article is to develop a holistic mathematical model and software implementation of continuous-time optimal control algorithms that ensure the construction of trajectories with minimal energy consumption, compliance with given spatial constraints and avoidance of collisions with obstacles. The task of the work is to ensure the accuracy, energy efficiency and safety of the end effector motion with three degrees of freedom in continuous time. To do this, it is necessary to formulate optimal control conditions, taking into account kinematic, dynamic and energy constraints, as well as the risks of approaching forbidden zones. An additional task is to create a software implementation that will allow for numerical modeling and visualization of trajectories. It is expected that the implemented approach will ensure control adaptability to environmental changes and increase the efficiency of robotic systems in production and service scenarios.

Research results

1. Mathematical representation of the construction of collaborative robot-manipulator gripping device trajectories taking into account constraints and optimal control actions in continuous time, based on the Pontryagin maximum principle. Pontryagin's maximum principle is one of the fundamental methods in optimal control theory that formalizes the necessary optimality conditions for continuous-time control problems [28]. It states that for optimal control there is a nonzero vector of adjoint variables (coordinates) such that the optimal trajectory of the system together with these variables maximizes the Hamiltonian function at each time instant on the admissible set of controls [29].

The Hamiltonian, or Hamiltonian function, depends on the state of the system, control actions, adjoint variables, and time, and serves as the generalized energy that the system is trying to optimize. This principle allows us to reduce the optimal control problem to the analysis of a system of ordinary differential equations with additional optimality conditions, which greatly simplifies the search for a solution in complex problems with constraints. The principle is a powerful tool in building mathematical models in many applied problems, in particular in the dynamic control of robotic systems [30, 31].

In the first step, it is necessary to formulate the optimal control problem, let the dynamic system be considered in the form of a controlled mechanical model of the manipulator within the framework of these studies:

$$\dot{x}(t) = f\left(x(t), u(t)\right), \quad x(t_0) = x_0,\ t \in \left[t_0, t_f\right], \quad (1)$$

where $x(t) \in \mathbb{R}^n$ – system state vector (e.g., joint coordinates, velocities); $u(t) \in \mathbb{R}^m$ – vector of control actions (e.g., moments of forces on the axes); x_0 – initial state; $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ – system dynamics function; t_f – final moment of time.

Let's choose the optimization goal - to minimize energy consumption or time:

$$J = \int_{t_0}^{t_f} L\left(x(t), u(t), t\right)\mathrm{dt} + \Phi\left(x\left(t_f\right)\right), \quad (2)$$

where J – objective function (optimization functional), which reflects the total "cost" or "benefit" of the control process over the entire time interval $[t_0, t_f]$, the goal is to find such a control action $u(t)$, which minimizes or maximizes this value; t_0 – the initial point in time from which the control process begins; t_f – the final point in time to which the process is considered can be fixed or variable; $x(t) \in \mathbb{R}^n$ – the system state vector at time t, which describes the current physical or logical state of the control object, its dynamics is given by a system of differential equations; $u(t) \in \mathbb{R}^m$ – vector of control actions (control influences), which can be changed in order to achieve the optimal result, they are imposed on the system externally; $L(x(t), u(t), t)$ – current (instantaneous) cost (benefit) function that determines the "cost" of the system being in the state $x(t)$ when using control $u(t)$ in time t, this integrand evaluates the quality of control throughout the process; $\Phi(x(t_f))$ – a final cost function that estimates the cost (or benefit) of the system being in the final state $x(t_f)$. It allows you to specify the desired result, for example, the minimum distance to the goal, the final energy, etc.

To use the Pontryagin maximum principle, the following Hamiltonian (or Hamilton function) is introduced. It is used in optimal control to formalize the conditions [32] under which the extremum of the control functional is achieved:

$$H\left(x, u, \lambda, t\right) = L\left(x, u, t\right) + \lambda^T f(x, u) \quad (3)$$

where $H(x, u, \lambda, t)$ – a scalar function (Hamiltonian) that combines instantaneous costs $L(x, u, t)$ and system dynamics $f(x, u)$ using a vector of conjugate variables λ, its maximum at each time point determines the optimality of control according to the Pontryagin maximum principle; $x(t) \in \mathbb{R}^n$ - system state vector at a point in time t, it describes the internal state of the control object (e.g. position, speed, temperature, etc.); $u(t) \in \mathbb{R}^m$ - vector of control actions, i.e. changes that are introduced into the system in order to achieve the desired result (e.g. force, voltage, engine speed, etc.); $\lambda \in \mathbb{R}^n$ - a vector of conjugate variables (Pontryagin multipliers), which is introduced as an auxiliary quantity. It is interpreted as a shadow price or gradient of the functional over states, i.e. it determines how much a change in each state variable affects the overall goal; $t \in \mathbb{R}$ - continuous time, a variable that reflects the current moment in an interval $[t_0, t_f]$; $L(x, u, t)$ - an instantaneous cost or benefit function that determines the local "cost" or "benefit" of being in a state x while control u in time t. This element is included in the integral in the functionality as a current assessment of the quality of the process; $f(x, u) \in \mathbb{R}^n$ - vector function of the system dynamics that determines how the state x changes under the control u influence, it defines a system of differential equations:

$$\dot{x}(t) = f(x(t), u(t)); \quad (4)$$

$\lambda^T f(x, u)$ - the dot product of the vector of adjoint variables and the dynamics vector, which represents the influence of the system dynamics on the functional through the change of states. This term effectively "weights" the influence of each state variable on the final goal.

Thus, the Hamiltonian combines information about how beneficial a particular control u is in the current state x of the system, taking into account both local costs and further evolution of the system. Its maximization at each time point determines the optimal trajectory and control actions. We propose the following conditions for the Pontryagin maximum [33]:

– state dynamics, specifies the evolution of the system state over time under the available control$u(t)$:

$$\dot{x}(t) = \frac{\partial H}{\partial \lambda} = f(x(t), u(t)), \quad (5)$$

where $\dot{x}(t) \in \mathbb{R}^n$ - derivative of the system state vector with respect to time t, that is, the rate of change of each state component, it describes how the system changes over time under control; $\frac{\partial H}{\partial \lambda}$ – partial derivative of the Hamiltonian with respect to the conjugate variables. It is equal to the vector function of the dynamics of the system $f(x(t), u(t))$, this reflects that the state dynamics directly depends on the current values of the states and controls, but does not depend on the conjugate variables themselves; $f(x(t), u(t)) \in \mathbb{R}^n$ - vector function of the system dynamics that specifies how states change $x(t)$ when using control $u(t)$. This is a functional description of the physical or logical process that underlies the control object; $x(t) \in \mathbb{R}^n$ - system state vector at time instant t. It contains all the variables that describe the current state of the control object (for example, the position of the robot manipulator joints, their speed, temperature, voltage, etc.); $u(t) \in \mathbb{R}^m$ - vector of control actions that affect the dynamics of states. The choice of control determines how the system changes over time.

– covector dynamics (backward), describes how shadow prices or "cost covectors" change in the backward direction of time (backward dynamics), which is important for determining optimal control actions:

$$\lambda(t) = -\frac{\partial H}{\partial x}, \quad (6)$$

where $\lambda(t) \in \mathbb{R}^n$ - vector of conjugate variables (covector), also known as the Pontryagin multiplier or shadow price, it plays the role of the gradient of the functional over the states of the system and characterizes the impact of changing each state component on the value of the quality criterion [34]; $\frac{\partial H}{\partial x} \in \mathbb{R}^n$ – partial derivative of the Hamiltonian with respect to the state x, it shows how the Hamiltonian (and therefore the optimization functional) changes when the state of the system changes.

– maximum condition:

$$u^* = arg \max_{u \in U} H\left(x(t), u, \lambda(t), t\right), \quad (7)$$

where: $u^*(t) \in U \subseteq \mathbb{R}^m$ – optimal control action at a point in time t, this is the control that provides the maximum of the Hamiltonian for fixed states and conjugate variables; $u \in U$ – the set of permissible controls, is a restriction on the control signals that can be applied to the system; $H(x(t), u, \lambda(t), t)$ – Hamiltonian, i.e. a function that combines the instantaneous benefit (through the function L) and the dynamics of the system (through the vector f).

Thus, condition (7) says that at each time step it is necessary to choose such control $u^*(t)$, which maximizes the Hamiltonian given the current state of the system and the adjoint variables. This is a key condition in optimal control problems for continuous systems and is a mandatory component in the application of the Pontryagin maximum principle.

– boundary conditions, specify the initial and final values for the state variables and adjoint variables, which allows the complete continuous-time optimal control problem to be formulated as a two-point boundary value problem:

$$x(t_0) = x_0;$$
$$\text{if } \Phi\left(x\left(t_f\right)\right) \text{ given, then } \lambda\left(t_f\right) = \frac{\partial \Phi}{\partial x\left(t_f\right)}, \quad (8)$$

where: $x(t_0) = x_0$ – initial condition for the state vector, it indicates that the state of the dynamical system at the time point t_0 is known and equal to the vector x_0; $\Phi(x(t_f))$ – terminal (final) function of the objective functional, it takes into account the significance of reaching a certain state at the end of the control interval t_f; $\lambda\left(t_f\right) = \frac{\partial \Phi}{\partial x\left(t_f\right)}$ – final condition for conjugate variables (covector), if the function Φ is given explicitly, then the covector at the moment t_f is calculated as the gradient of the terminal function over the state at the end of the control.

Thus, these boundary conditions provide a complete statement of the optimization problem: the initial state point is known $x(t_0)$, and at the end of the state trajectory, if the objective function contains $\Phi(x(t_f))$, a boundary condition is given for the conjugate vector $\lambda(t_f)$. This allows us to solve the problem in the form of two coupled integrations - forward for the state $x(t)$ and back for the covector $\lambda(t)$.

2. Development of a program for modeling and calculating the optimal trajectories of the gripping device of a collaborative robot-manipulator based on the Pontryagin maximum principle. The choice of the Python programming language for developing a program for modeling and calculating the optimal trajectory of the gripper of a collaborative robot-manipulator based on the Pontryagin maximum principle is due to its simplicity, high readability, powerful scientific and computational ecosystem and active community [34–36]. The NumPy, SciPy and Matplotlib libraries provide convenient tools for implementing numerical methods, processing data sets, optimizing and visualizing results in three-dimensional space [37–40]. Using the PyCharm 2022.2.3 development environment allows you to effectively organize the project structure, facilitates code debugging, autocompletion and version control, which is especially important for complex algorithms of dynamic programming and optimal control [41]. In addition, Python is well suited for integration with robotics simulators, making it an ideal tool for building and testing collaborative robot models [42–45]. The general algorithm of the program for modeling and calculating the optimal trajectories of the gripper of a collaborative robot-manipulator based on the Pontryagin maximum principle is presented in Fig. 1.

Fragments of the software implementation of the program for modeling and calculating the optimal trajectories of the gripping device of a collaborative robot-manipulator based on the Pontryagin maximum principle are given below:

```
# Manipulator parameters
link_lengths = [0.4, 0.3, 0.3]  # three-link lengths
m_eff = 2.0  # mass of effector (kg)
t0, tf = 0, 10
N = 100
dt = (tf - t0) / N
time = np.linspace(t0, tf, N)
```

This code fragment specifies the main parameters of the three-link manipulator, in particular the lengths of each of the three links and the mass of the gripping device (effector), which affect the kinematics and dynamics of the movement. The simulation time interval is also defined - from the initial moment $t_0 = 0$ to the final $t_f = 10$ seconds, number of discrete steps $N = 100$ and the integration step dt is calculated, which is necessary for numerically solving the equations of motion using the Euler method.

```
def collision_penalty(x_path):
    penalty = 0.0
    for obs in obstacles:
        d = np.linalg.norm(x_path - obs["center"], axis=1)
        violation = np.maximum(0, obs["radius"] - d)
        penalty += np.sum(violation ** 2)
    return penalty * 1e5
```

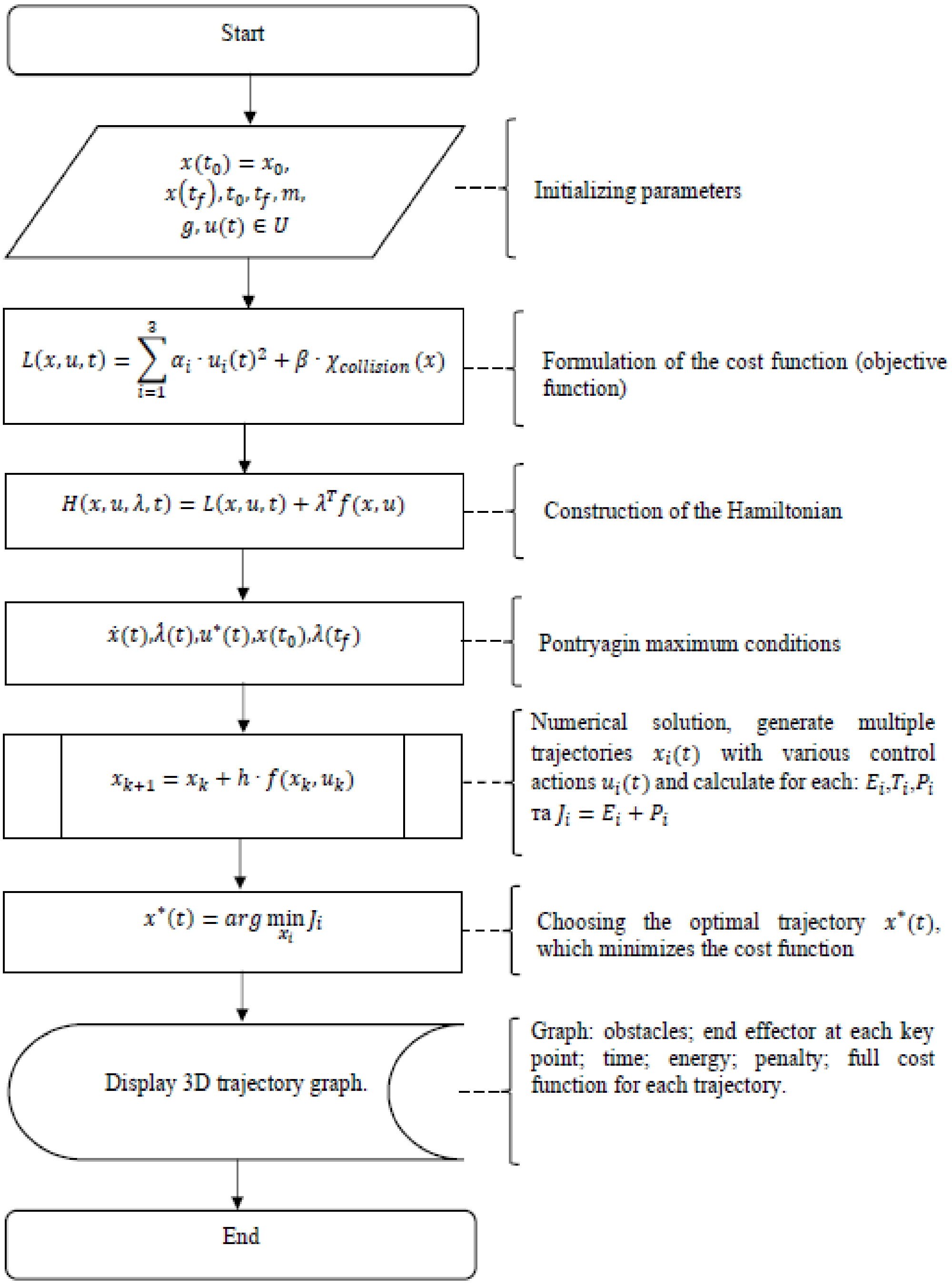

Fig. 1. General view of the algorithm of the program
for modeling and calculating the optimal trajectories of the gripping device
of a collaborative robot-manipulator based on the Pontryagin maximum principle

This code fragment implements the function of calculating the penalty for collisions of the gripper with obstacles while moving along a given trajectory.

For each obstacle, a violation is calculated (if the trajectory point is inside the radius of the obstacle), and the total penalty is formed as the sum of the squares of these violations, multiplied by a large coefficient (1e5) to make collisions energetically and algorithmically disadvantageous during optimization:

```
def compute_energy(u_flat):
    u = u_flat.reshape((N, 3))
    a_squared = np.sum(u ** 2, axis=1)
    return m_eff * np.sum(a_squared) * dt
```

The presented code fragment calculates the approximate energy costs for the movement of the manipulator's gripping device along a given trajectory. It deploys the vector of control actions $u(t)$, finds the square of the accelerations at each step, sums them, and multiplies by the mass of the effectorm_{eff} and time step dt, modeling energy consumption by analogy with kinetic energy:

```
def total_cost(u_flat):
    u = u_flat.reshape((N, 3))
    v = np.cumsum(u, axis=0) * dt
    x = np.cumsum(v, axis=0) * dt + x0
    energy = compute_energy(u_flat)
```

```
    penalty = collision_penalty(x)
    return energy + penalty
```

The code snippet implements a full trajectory cost function that combines two key criteria: the energy cost of movement and the penalty for potential collisions with obstacles. It calculates the position of the effector by double integrating the acceleration (over the velocity and coordinates), and then returns the sum of the energy cost and penalty as a general optimization function to be minimized.

```
def end_position_constraint(u_flat):
    u = u_flat.reshape((N, 3))
    v = np.cumsum(u, axis=0) * dt
    x = np.cumsum(v, axis=0) * dt + x0
    return x[-1] – xf
```

This code snippet defines a constraint on the final position of the effector, which is used in trajectory optimization. It calculates the endpoint of the trajectory $x[-1]$ by double integration of the control accelerations and compares it with the desired final position x_f; the difference is returned as a constraint vector that must be zero to reach the goal point.

```
def forward_kinematics_3link(end_effector_pos):
  base = np.array([0, 0, 0])
  vec = end_effector_pos / np.linalg.norm(end_effector_pos)
  joint1 = base + link_lengths[0] * vec
    joint2 = joint1 + link_lengths[1] * vec
    joint3 = joint2 + link_lengths[2] * vec
    return np.array([base, joint1, joint2, joint3])
```

Rodu implements a direct kinematics model for a three-link manipulator oriented along the direction to a given effector position. It determines the position of the base and each joint by sequentially adding the vectors of the links along the normalized direction to the effector, which allows us to visualize the position of the manipulator in 3D space according to the endpoint of the trajectory.

3. Modeling the optimal trajectory of the gripping device of a collaborative robot-manipulator and analyzing the results obtained. To verify the correctness and adequacy of the proposed mathematical models and expressions on the basis of which the modeling program was developed, it is proposed to conduct an experiment to simulate the movement of a collaborative robot-manipulator, the following numerical values of the input parameters are used: the lengths of the three links of the manipulator are 0.4, 0.3 and 0.3 meters, respectively, which determines its constructive configuration.

The mass of the gripping device (effector) is 2.0 kg, which is necessary to calculate the energy consumption during movement. The initial moment of the modeling time is 0 seconds, the final moment is 10 seconds, and the number of discretization steps is set to 100, which gives a time step dt=0.1 seconds. The initial position of the effector is at the point with coordinates (0, 0, 0), and the goal position is at the point (1, 1, 1). Two obstacles are modeled in the workspace: the first with a center at the point (0.5, 0.5, 0.5) and a radius of 0.2 meters, and the second with a center at the point (0.3, 0.7, 0.6) and a radius of 0.15 meters, which affect the choice of the optimal trajectory of the effector movement.

The results of modeling the movement of a collaborative robot-manipulator with given input data are presented in Fig. 2.

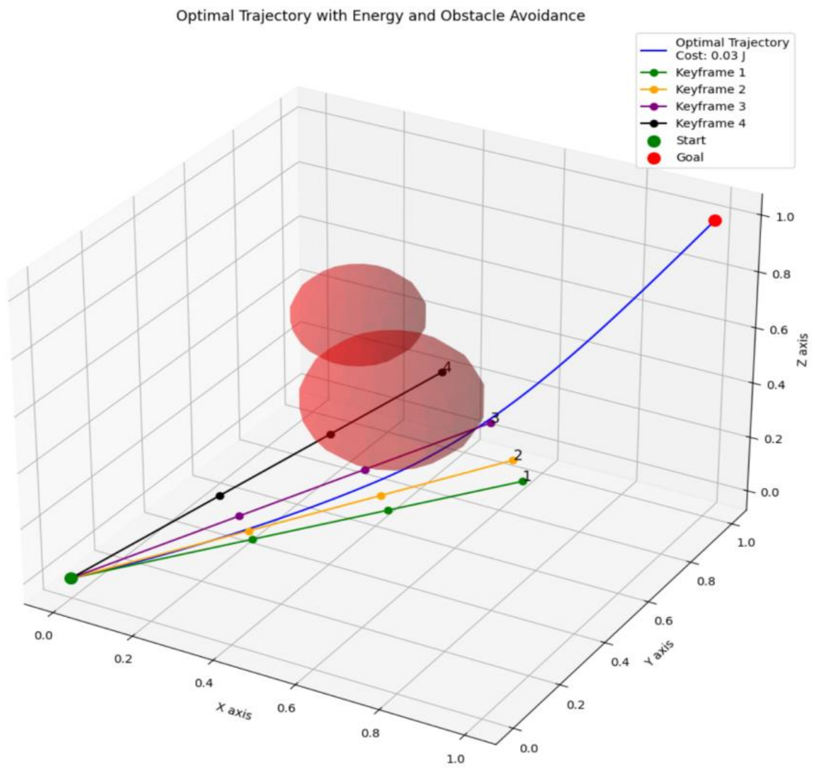

Fig. 2. Graph of the results of modeling the optimal trajectory of the gripping device of a collaborative robot manipulator

Fig. 2 presents the results of modeling the optimal trajectory of the manipulator, which takes into account energy consumption (0.03 J) and obstacle avoidance. The trajectory consists of four key frames, which indicates a step-by-step approach to movement from the initial to the final point. The smoothness of the line indicates the absence of sharp changes in direction, which ensures minimal load on the structure and energy efficiency. High accuracy of achieving the goal (Goal) confirms the effectiveness of the planning algorithm. The low cost of the trajectory (0.03 J) indicates the optimization of energy consumption, which is critical for collaborative robots. From which it is possible to draw the following conclusion: the proposed method provides effective, safe and economical movement of the manipulator in a space with obstacles. Fig. 3 shows the results of calculating the selected tractor of the end effector movement and 4 key frames (tractors).

```
◆ KEYFRAME 1: t = 0.00 s
  End-effector Position: [0.00044561
  Speed: 0.0060 m/s
  Energy Used: 0.0007 J
  Penalty: 0.0000
  Cost (Energy + Penalty): 0.0007
```

a – key frame №1

```
◆ KEYFRAME 2: t = 3.33 s
  End-effector Position: [0.22622542
  Speed: 0.1645 m/s
  Energy Used: 0.0165 J
  Penalty: 0.0000
  Cost (Energy + Penalty): 0.0165
```

b – key frame №2

```
◆ KEYFRAME 3: t = 6.67 s
  End-effector Position: [0.65184427
  Speed: 0.2227 m/s
  Energy Used: 0.0238 J
  Penalty: 0.0000
  Cost (Energy + Penalty): 0.0238
```

c – key frame №3

```
◆ KEYFRAME 4: t = 10.00 s
  End-effector Position: [1. 1. 1.]
  Speed: 0.2581 m/s
  Energy Used: 0.0270 J
  Penalty: 0.0000
  Cost (Energy + Penalty): 0.0270
```

d – key frame №4

Fig. 3. Results of simulation calculations and finding the tractor motion of the end effector and 4 key frames (tractors)

Analysis of the simulation results (Fig. 3) shows the high efficiency of the trajectory planning algorithm. The total energy consumption was only 0.0270 J, which indicates the economy of the manipulator movement. The absence of penalties for obstacles (0.0000) confirms the successful avoidance of collisions. In the first key frame (Fig. 3a) (t = 0.00 s), the speed was minimal (0.0060 m/s*), and the energy consumption was insignificant (0.0007 J), which is typical for the beginning of the movement. In the second frame (Fig. 3b) (t = 3.33 s), the speed increased to 0.1645 m/s, and the energy consumption to 0.0165 J, which corresponds to the active phase of movement. The third frame (Fig. 3c) (t = 6.67 s) demonstrates a further increase in speed (0.2227 m/s) and energy consumption (0.0238 J), indicating a stable acceleration. In the last frame (Fig. 3d) (t = 10.00 s), the manipulator reached the target position ([1, 1, 1]) with a maximum speed (0.2581 m/s) and full energy consumption (0.0270 J). This allows us to draw the following conclusion, the algorithm provided a smooth, energy-efficient and safe collision-free motion, which is ideal for collaborative robots. The results confirm the accuracy of the modeling and the practical suitability of the method for real-world applications.

Conclusions

As a result of the research, a mathematical description of the construction of the trajectories of the gripping device of a collaborative robot-manipulator was formed, taking into account physical constraints, dynamic characteristics and optimal control actions in continuous time. The proposed model is based on analytical consideration of the kinematic and dynamic properties of the manipulator, including boundary conditions, speed and acceleration constraints. As part of the work, a control structure was implemented that allows forming smooth, physically achievable and optimal trajectories of the end effector according to a given criterion. The developed algorithms provide flexible response to changing task conditions, adaptation to the load and taking into account the presence of obstacles in the working area. Special attention is paid to methods of feedback and trajectory correction in real time while maintaining system stability. Continuous-time simulation modeling was carried out using the Python environment, which demonstrated the consistency between analytical calculations and the behavior of the system in the model. The results confirm that the proposed mathematical software allows for effective control of the movement of the gripping device in a collaborative environment, minimizing risks and ensuring positioning accuracy. The developed model can be used as a basis for building adaptive controllers with flexible trajectory reconstruction when changing external conditions. In the future, it is recommended to expand the functionality of the model taking into account reinforcement learning methods, multi-link manipulation systems, as well as verification of the results obtained on a physical robotic stand to increase the applied reliability.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

Acknowledgment

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Математичне моделювання побудови траєкторій руху захватного пристрою колаборативного робота

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Анотація. Об'єктом дослідження є процес побудови та аналізу траєкторій руху захоплювального пристрою колаборативного робота-маніпулятора в умовах просторових обмежень та наявності перешкод у динамічному середовищі. **Предметом дослідження** є математичні моделі, алгоритмічне та програмне забезпечення для моделювання оптимального руху кінцевого ефектора маніпулятора з урахуванням кінематичних, динамічних та енергетичних обмежень. **Метою роботи** побудова траєкторій руху захватного пристрою колаборативного робота з урахуванням обмежень та оптимальних керуючих дій у безперервному часі, що забезпечують побудову траєкторій із мінімальними енерговитратами, дотриманням заданих просторових обмежень та уникненням зіткнень із перешкодами. **Методологія дослідження** ґрунтується на застосуванні принципу максимуму Понтрягіна для формування умов оптимального керування та побудови системи диференційних рівнянь із граничними умовами. Для кількісної оцінки енерговитрат та врахування штрафних санкцій за наближення до заборонених зон розроблено спеціальний функціонал вартості. Чисельне розв'язання задачі реалізовано методом Ейлера, а оптимізацію параметрів траєкторії з фіксованими кінцевими координатами ефектора — методом найменших квадратів із обмеженнями. Для візуалізації результатів використано мову програмування Python та бібліотеку Matplotlib. **У результаті дослідження** отримано оптимальні траєкторії руху захоплювального пристрою, які забезпечують уникнення зіткнень, відповідність просторовим обмеженням та зменшення енергоспоживання при досягненні заданих кінцевих положень ефектора. Моделювання підтвердило ефективність розробленої методики та її стійкість до зміни параметрів середовища. **Висновки** дослідження свідчать, що запропонований підхід дозволяє комплексно вирішувати завдання планування руху колаборативних роботів у режимі оптимального керування з урахуванням обмежень. Отримані результати можуть бути застосовані у виробничих системах Індустрії 5.0, роботизованих сервісних комплексах, автоматизованих складських системах та роботах, що взаємодіють із людиною в обмеженому просторі.

Ключові слова: колаборативний робот; робот-маніпулятор; траєкторія руху робота; захоплювальний пристрій; математичне моделювання; принцип максимуму Понтрягіна.