

Lev Raskin, Oksana Sira, Yurii Parfeniuk

National Technical University "Kharkiv Polytechnic Institute, Kharkiv, Ukraine

## SELECTION OF THE OPTIMUM ROUTE IN AN EXTENDED TRANSPORTATION NETWORK UNDER UNCERTAINTY

**Abstract.** Relevance. For a given values set of extensive transport network sections lengths an exact method has been developed for finding optimal routes. The method provides an approximate solution when the initial data - are random variables with known distribution laws, as well as if these data are not clearly specified. For a special case with a normal distribution of the numerical characteristics of the network, solution is brought to the final results. Method. An exact method of deterministic routing is proposed, which gives an approximate solution in case of random initial data. The method is extended to the case when the initial data are described in theory of fuzzy sets terms. The problem of stability assessing of solutions to problems of control the theory under conditions of uncertainty of initial data is considered. Results. A method of optimal routes finding is proposed when the initial data are deterministic or random variables with known distribution densities. A particular case of a probabilistic - theoretical description of the initial data is considered when can be obtained a simple solution of problem. Proposed method for obtaining an approximate solution in the general case for arbitrary distribution densities of random initial data. The situation is common when the initial data are not clearly defined. A simple computational procedure proposed for obtaining a solution. A method for stability assessing of solutions to control problems adopted under conditions of uncertainty in the initial data, is considered.

**Keywords:** transport network; optimal route; initial data - random or fuzzy numbers; stability of solutions to control problems.

### Introduction

The problem of optimal route finding represents an integral part of a more general, so-called, transport problem of linear programming [1].

**Formulation of the problem.** The problem statement is as follows: there are some given points of a certain product manufacturing  $A_1, A_2, \dots, A_m$  and points of this product consumption  $B_1, B_2, \dots, B_n$ . For each manufacturing point  $A_i$  there it is given  $a_i$  volume of manufacturing,  $i = 1, 2, \dots, m$  and for each point  $B_j$  of consumption  $b_j$  - the amount of consumption is also expected. So it is supposed to be known of product transportation routes from producers to consumers and there it must be set the appropriate matrix of values for the average cost of a product unit transportation  $i = 1, 2, \dots, m, j = 1, 2, \dots, n$ . It is required to find a matrix  $X = (x_{ij})$  of values for planned transportation volumes that minimizes

$$L(x) = \sum_{i=1}^m \sum_{j=1}^n C_{ij} x_{ij}$$

and satisfies the next constraints:

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = \overline{1, m}, \quad \sum_{i=1}^m x_{ij} = b_j, \quad j = \overline{1, n},$$

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j.$$

**Materials and methods.** It is clear that the total average cost of transportation, which determines efficiency of plan X, depends upon a set of values  $(C_{ij})$ , that are determined by routes connecting the points of manufacturing and consumption. The problem of forming a route for any pair of manufacturers -

consumer is one of the variants in general problem of the scheduling theory and for the particular case under consideration was mentioned in [2-4].

However, in the general case for distributed transport systems of high dimension, it was not considered. In this regard, the task of constructing optimal routes is urgent. The position of manufacturing points and product consumption is given. Each pair of these points is connected by a route made up of a set of sections specified by the points of their start and end.

Formally, it is natural to describe the problem model using a directed graph, the vertices of which correspond to route intermediate points, and to the arcs - the sections between these points. The length of each arc determines a quantitative measure efficiency of the corresponding section using in the desired route connecting manufacturing point and point of consumption.

The task is to find sequence of passage the selected sections set, total measure efficiency of which determines the best (in the chosen sense) route. If all the points are numbered, then it is convenient to specify each of the sections set with the numbers (i, j) points at beginning and at the end of this section. In this case, the triple of numbers (i, k, j) defines a pair of sequentially connected sections (i, k) and (k, j). If the intermediate point k between points i, j is not the only possible one, then the problem of its best choice arises.

Supposed that as a measure of the usage efficiency in the route is selected by average value of a cargo unit transportation through this section and the predetermined values set of that measure in a matrix  $C = C_{ij}$ . This value  $C_{ij}$  is defined by following rule: the number gives a measure of section efficiency with a starting point  $i$  and endpoint  $j$ ,  $C_{ii} = 0$  and  $C_{ij} = M$  ( $M$  - large number), if items  $i$  and  $j$  belong to different areas. Then the best intermediate point  $k^*$  between points  $i$  and  $j$  can be chosen using the relation

$$C_{ik}^* + C_{kj}^* = \min_{k \in K} \{C_{ik} + C_{kj}\}, \quad (1)$$

where  $K$  – quantity of intermediate points between  $i$  and  $j$ .

Operation (1) introduced for two separated adjacent segments can be summarized as a sequence of portions of arbitrary length. For this purpose, we use the matrix commutation operation [5]. For two arbitrary matrices  $A$  and  $B$  of dimension  $n \times n$ , we introduce the matrix commutation operation  $\otimes$  by the formula  $C = A \otimes B$ ,

$$C_{ij} = \min_k (C_{ik} + C_{kj}), \quad i = \overline{1, m}, j = \overline{1, n}, k = \overline{1, n}. \quad (2)$$

For an arbitrary system of  $n$  points, we define matrix  $C^{(1)} = (C_{ij}^{(1)})$  according to the rule formulated above. Then we calculate

$$C^{(2)} = C^{(1)} \otimes C^{(1)} \quad (3)$$

The elements of  $C^{(2)}$  matrix, calculated by the formula (2) define length of the most effective in accordance with the selected measure for Two-step paths between all pairs of points  $(i, j)$ .

Let us look at a simple example. Let the route between point of manufacturing and consumption point pass through one of three intermediate points (Fig. 1).

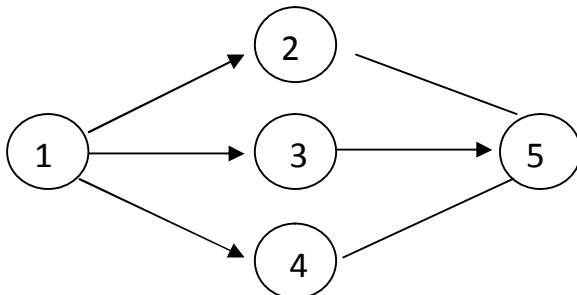


Fig. 1. Route system from point 1 to point 5

The values of average transportation cost for transport network segments, shown in Fig. 1, will be introduced into  $C^{(1)}$  matrix:

$$C^{(1)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{matrix} \begin{matrix} 0 & 4 & 3 & 5 & M \end{matrix} \\ \begin{matrix} M & 0 & M & M & 6 \end{matrix} \\ \begin{matrix} M & M & 0 & M & 8 \end{matrix} \\ \begin{matrix} M & M & M & 0 & 7 \end{matrix} \\ \begin{matrix} M & M & M & M & 0 \end{matrix} \end{matrix} \end{matrix}$$

Let us perform operation (3) taking into account (2). Wherein:

$$C_{11}^{(2)} = \min \left\{ \begin{matrix} (C_{11}^{(1)} + C_{11}^{(1)}), (C_{12}^{(1)} + C_{21}^{(1)}), (C_{13}^{(1)} + C_{31}^{(1)}), \\ (C_{14}^{(1)} + C_{41}^{(1)}), (C_{15}^{(1)} + C_{51}^{(1)}) \end{matrix} \right\} = \min \left\{ \begin{matrix} (0+0), (4+M), (3+M), \\ (5+M), (M+M) \end{matrix} \right\} = 0;$$

$$C_{12}^{(2)} = \min \left\{ \begin{matrix} (C_{11}^{(1)} + C_{12}^{(1)}), (C_{12}^{(1)} + C_{22}^{(1)}), (C_{13}^{(1)} + C_{32}^{(1)}), \\ (C_{14}^{(1)} + C_{42}^{(1)}), (C_{15}^{(1)} + C_{52}^{(1)}) \end{matrix} \right\} = \min \left\{ \begin{matrix} (0+4), (4+0), (3+M), \\ (5+M), (M+M) \end{matrix} \right\} = 4;$$

$$C_{13}^{(2)} = \min \left\{ \begin{matrix} (C_{11}^{(1)} + C_{13}^{(1)}), (C_{12}^{(1)} + C_{23}^{(1)}), (C_{13}^{(1)} + C_{33}^{(1)}), \\ (C_{14}^{(1)} + C_{43}^{(1)}), (C_{15}^{(1)} + C_{53}^{(1)}) \end{matrix} \right\} = \min \left\{ \begin{matrix} (0+3), (4+M), (3+0), \\ (5+M), (M+M) \end{matrix} \right\} = 3;$$

$$C_{14}^{(2)} = \min \left\{ \begin{matrix} (C_{11}^{(1)} + C_{14}^{(1)}), (C_{12}^{(1)} + C_{24}^{(1)}), (C_{13}^{(1)} + C_{34}^{(1)}), \\ (C_{14}^{(1)} + C_{44}^{(1)}), (C_{15}^{(1)} + C_{54}^{(1)}) \end{matrix} \right\} = \min \left\{ \begin{matrix} (0+5), (4+M), (3+M), \\ (5+0), (M+M) \end{matrix} \right\} = 5;$$

$$C_{15}^{(2)} = \min \left\{ \begin{matrix} (C_{11}^{(1)} + C_{15}^{(1)}), (C_{12}^{(1)} + C_{25}^{(1)}), (C_{13}^{(1)} + C_{35}^{(1)}), \\ (C_{14}^{(1)} + C_{45}^{(1)}), (C_{15}^{(1)} + C_{55}^{(1)}) \end{matrix} \right\} = \min \left\{ \begin{matrix} (0+M), (4+6), (3+8), \\ (5+7), (M+0) \end{matrix} \right\} = 10;$$

$$C_{21}^{(2)} = C_{23}^{(2)} = C_{24}^{(2)} = M, \quad C_{25}^{(2)} = 6;$$

$$C_{31}^{(2)} = C_{32}^{(2)} = C_{34}^{(2)} = M, \quad C_{35}^{(2)} = 8;$$

$$C_{41}^{(2)} = C_{42}^{(2)} = C_{43}^{(2)} = M, \quad C_{45}^{(2)} = 7;$$

Let us summarize the results obtained in a matrix  $C^{(2)}$ :

$$C^{(2)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{matrix} \begin{matrix} 0 & 4 & 3 & 5 & 10 \end{matrix} \\ \begin{matrix} M & 0 & 0 & M & 6 \end{matrix} \\ \begin{matrix} M & M & 0 & M & 8 \end{matrix} \\ \begin{matrix} M & M & M & 0 & 7 \end{matrix} \\ \begin{matrix} M & M & M & M & 0 \end{matrix} \end{matrix} \end{matrix}$$

Thus, the optimal route from point 1 to the point 5 passes through an intermediate point 2 and has a measure of effectiveness - 10.

The introduced commutation operation can be used to find optimal route of arbitrary length by recurrent calculating in sequence of matrices:

$$C^{(3)} = C^{(2)} \otimes C^{(1)}, \quad C^{(k+1)} = C^{(k)} \otimes C^{(1)}. \quad (4)$$

The formulated problem of optimal route finding becomes much more complicated if the initial data has uncertainty. The solution of problem under these conditions is important for theoretical and practical interest.

The aim of work is to construct a computational procedure for solving the problem of finding optimal

routes under conditions when the initial data have random variables with known distribution laws. In accordance to this, we will assume that the cost of transportation for each section  $C_{ij}$  is a random variable with a known distribution density  $\phi_{ij}(C_{ij})$ .

**Main results**

Let us give some simplest example of possible solving problem technology for finding optimal route for a transport network, shown in Fig. 1. In this task let us assume to be known the densities of distribution  $\phi_{ij}(C_{ij})$ ,  $i = 1, 2, 3, 4$ ;  $j = 2, 3, 4, 5$ . Then with basics on well-known rules of probability theory, we find the density distribution  $\phi_{1k5}(C_{1k5})$ ,  $C_{1k5} = C_{1k} + C_{k5}$ ,  $k = 2, 3, 4$ . As the optimal criterion of route using some probability that is a random value of passing on this route exceeds the threshold, the allowable value, we will calculate

$$P[(C_{1k5} = C_{1k} + C_{k5}) > C_{\Pi}] = \int_{C_{\Pi}}^{\infty} \phi_{1k5}(C_{1k5}) dC_{1k5}, \quad K = 2, 3, 4. \tag{5}$$

The best route will be that one for which probability of exceeding the threshold will be the least.

This technology can be difficult to implement even in this simplest particular case by the need to find an analytical expression for the density  $\phi_{ij}(C_{ij})$  by taking integral

$$\phi_{ikj}(C_{ikj}) = \int_0^{\infty} \phi_{ik}(C_{ik}) \cdot \phi_{kj}(C_{ikj} - C_{ik}) dC_{ik}$$

with subsequent use in (5). In this case, analytical expression for the compositional distribution density  $\phi_{ikj}(C_{ikj})$  will be more complicated than expressions for the composition elements  $\phi_{ik}(C_{ik})$  and  $\phi_{kj}(C_{kj})$ . This means that the recurrent solution of a more general problem in accordance with technology (4) at each subsequent step will be more difficult than at previous one.

Note, however, that in some special cases a simple solution to the problem is possible if, for example, the random variables are distributed according to normal distribution law. Suppose that in the example considered above, random values of transportation costs are distributed normally and the corresponding densities have the form:

$$\begin{aligned} \phi_{12}(C_{12}) &= \frac{1}{\sqrt{2\pi} \cdot 2} \exp\left\{-\frac{(C_{12}-4)^2}{8}\right\}, \\ \phi_{13}(C_{13}) &= \frac{1}{\sqrt{2\pi} \cdot 3} \exp\left\{-\frac{(C_{13}-3)^2}{18}\right\}, \\ \phi_{14}(C_{14}) &= \frac{1}{\sqrt{2\pi} \cdot 9} \exp\left\{-\frac{(C_{14}-5)^2}{2}\right\}, \\ \phi_{25}(C_{25}) &= \frac{1}{\sqrt{2\pi} \cdot 3} \exp\left\{-\frac{(C_{25}-6)^2}{18}\right\}, \end{aligned}$$

$$\begin{aligned} \phi_{35}(C_{35}) &= \frac{1}{\sqrt{2\pi} \cdot 4} \exp\left\{-\frac{(C_{35}-8)^2}{32}\right\}, \\ \phi_{45}(C_{45}) &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(C_{45}-7)^2}{2}\right\}. \end{aligned}$$

Let define expression for density distribution of composite random variables  $C_{ijk}$ .

$$\begin{aligned} \phi_{125}(C_{125}) &= \phi_{125}(C_{12} + C_{25}) = \\ &= \frac{1}{\sqrt{2\pi} (4+9)^{1/2}} \exp\left\{-\frac{[C_{125} - (4+6)]^2}{2(4+9)}\right\} = \\ &= \frac{1}{\sqrt{2\pi} \cdot 3.6} \exp\left\{-\frac{(C_{125}-10)^2}{26}\right\}; \end{aligned}$$

$$\begin{aligned} \phi_{135}(C_{135}) &= \phi(C_{13} + C_{35}) = \\ &= \frac{1}{\sqrt{2\pi} (9+16)^{1/2}} \exp\left\{-\frac{[C_{135} - (3+8)]^2}{2(9+16)}\right\} = \\ &= \frac{1}{\sqrt{2\pi} \cdot 5} \exp\left\{-\frac{(C_{135}-11)^2}{50}\right\}; \end{aligned}$$

$$\begin{aligned} \phi_{145}(C_{145}) &= \phi_{145}(C_{14} + C_{45}) = \\ &= \frac{1}{\sqrt{2\pi} (1+1)^{1/2}} \exp\left\{-\frac{[C_{145} - (5+7)]^2}{2(1+1)}\right\} = \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(C_{145}-12)^2}{4}\right\}. \end{aligned}$$

Let us now calculate values of optimality criterion for routes using (5). Let us choose the threshold value for cost of transportation equal to maximum of the average values of composite costs with a certain weight coefficient  $\alpha$ . Then

$$C_{kp} = \alpha \cdot \max(10; 11; 12) = \alpha \cdot 12 = 13.8; \quad \alpha=1.15$$

$$\begin{aligned} P(C_{ikj} > C_{\Pi}) &= \int_{C_{\Pi}}^{\infty} \phi_{ikj}(C_{ikj}) dC_{ikj} = \\ &= \int_{C_{\Pi}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{ikj}} e^{-\frac{(C_{ikj}-\bar{C}_{ikj})^2}{2\sigma_{ikj}^2}} dC_{ikj} = \int_{(C_{\Pi}-\bar{C}_{ikj})/\sigma_{ikj}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \int_{a_{ikj}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du. \end{aligned} \tag{6}$$

The best route corresponds to minimum value of probability of exceeding the critical value, that is, maximum of lower limit values  $a_{ikj}$  in integral (6) for different routes. Thus, the value  $a_{ikj}^* = \max_k \{a_{ikj}\}$  becomes a criterion for route choosing. Let us calculate  $a_{ikj}$  value:

$$a_{125} = \frac{C_{\Pi} - \bar{C}_{125}}{\sigma_{125}} = \frac{13.8 - 10}{13^{0.5}} = \frac{3.8}{3.61} = 1.052,$$

$$a_{135} = \frac{C_{II} - \bar{C}_{135}}{\sigma_{135}} = \frac{13.8 - 11}{5} = \frac{2.8}{5} = 0.56,$$

$$a_{145} = \frac{C_{II} - \bar{C}_{145}}{\sigma_{145}} = \frac{13.8 - 12}{2^{0.5}} = \frac{1.8}{1.41} = 1.277,$$

$$a_{15}^* = \max_k \{1.052; 0.56; 1.277\} = a_{145}.$$

Thus, the route through intermediate point 4, which has worst value of average transportation cost, turned out to be the best due to minimum level of uncertainty in estimating transportation cost.

It is clear that in the particular case considered, the execution of procedure (2) - (3) for choosing best route does not cause any difficulties. An approximate (empirical) solution to problem in the general case can be obtained if in some natural way the method for calculating criterion for choosing a route  $a_{ikj}^*$  is simplified.

The value of this criterion is maximum if the values of transportation average cost and variance of this value are minimal. In this regard, let us choose  $\eta_{ikj} = C_{ikj} \cdot \sigma_{ikj}$  value. The best route ( $ikj$ ) corresponds to minimum value  $\eta_{ikj}$ . In the considered example, we have

$$\eta_{125} = C_{125} \cdot \sigma_{125} = 10 \cdot 3.61 = 36.1;$$

$$\eta_{135} = C_{135} \cdot \sigma_{135} = 11 \cdot 5 = 55;$$

$$\eta_{145} = C_{145} \cdot \sigma_{145} = 12 \cdot 1.41 = 16.2.$$

Thus, the route (1-4-5) is the best again. The possibility of practical use of this criterion is determined by the simplicity of calculating the mean values and variances of random variables for any density of their distribution. Finally, note that introduction of a new criterion  $\eta$  expands the possibilities of using proposed technology for optimal route choosing, to the case when uncertainty of initial data is described in terms of the fuzzy sets theory [5-8]. Indeed, let the membership function  $\mu(C)$  be used to describe the fuzzy value of transportation cost  $C$ . Let us introduce the function [9]

$$f(c) = \mu(C) / \int_0^{\infty} \mu(C) dC.$$

This function has all the properties distribution density of random variable: it is not negative and

$$\int_0^{\infty} f(C) dC = 1.$$

This function now calculates the expected value in usual way (mathematical expectation analogue) and variation (analogue of the standard deviation) which is used to calculate  $\eta$  criterion, after which described technology of finding optimal route is implemented.

Let us consider another important problem that usually arises when solving control problems. The modern theory of management traditionally assumptions that there is an uncertainty in the description of control system  $u$  environment in which the system operates.

Taking this uncertainty into account is necessary when solving the problem of optimal control itself and

when assessing its quality. In this case, question of how initial uncertainty affects on solution correctness of control problem and on possibility of retaining this solution in the presence of uncertainty, i.e on solution stability.

Various definitions of stochastic stability concept are well-known. A solution is called stable by probability if it is possible to find such a level of initial uncertainty at which the probability of deviation from decision does not exceed the given one [10]. A solution is called stable in terms mathematical expectation of a norm if it is possible to find such a level of perturbation at which the probability of deviations of the norm from the adopted decision does not exceed a given value [11]. In accordance with this, property of control system to develop correct solution and keep it in a certain range of random distortions of input information is called the stochastic stability of the system [12].

The problem of assessing stochastic stability has principal importance when solving optimal control problems under conditions of uncertainty in initial data. The criteria and method for solving this problem essentially depends on whether the set of solutions  $Q$  is continuous or discrete. In cases where this set is continuous, then the following criteria is usually used to assess stochastic stability [13 – 15]:

a) probability that the deviation from norm of objective function from its optimal value does not exceed the given one;

b) distortion degree of optimal solution;

c) deviation variance of objective function numerical value from the optimal value obtained in absence of uncertainty.

If the set of solutions is discrete, then natural criterion for assessing stability is probability of optimal solution distortion in the presence of uncertainty. Considering a method for solving the problem in this case, let us introduce the necessary notation.

Let a discrete set  $Q$  consist of  $n$  elements, that is  $Q = \{q_1, q_2, \dots, q_n\}$ . Let  $\Omega$  be a multidimensional continuous factorial space, each point of which corresponds to a vector  $w$ .

The numerical values of vector component is determined and the state of environment in system at a given time.

The decision-making algorithm is a certain operator  $A$  that maps points in the space  $\Omega$  to the elements of decisions  $Q$  set. Thus, a specific vector of environment states and system  $w_i$  corresponds to the decision  $q_i$  determined by rule:

$$q_i = A(w_i), \quad w_i \in \Omega, \quad q_i \in Q, \quad Q, i = 1, 2, \dots, n.$$

In accordance with this, space  $\Omega$  is divided into  $n$  domains  $\Omega_1, \Omega_2, \dots, \Omega_n$  so

$$\Omega_i = \{w : w \in \Omega, A(w) = q_i\} \quad i = 1, 2, \dots, n,$$

$$\cup \Omega_i = \Omega, \quad \cap \Omega_i = \emptyset.$$

Let  $q_{i0}$  be the optimal solution corresponding to current situation. Then probability of this optimal

decision will be made in presence of initial information uncertainty  $P_{i_0}$  is equal to

hitting in vector  $\hat{w}$  probability, determining state of the environment and system, in this situation, into range  $\Omega_{q_{i_0}}$ .

In this case, if  $f(\hat{w})$  - multidimensional distribution density of  $\hat{w}$  vector, then

$$P_{q_{i_0}} = \int_{\Omega_{q_{i_0}}} f(\hat{w}) d\hat{w}.$$

It follows that problem of assessing stability is reduced to solving the following two problems: finding an analytical description of range  $\Omega_i$  boundaries and probability calculating of vector  $w$  hit probabilities inside each of the ranges.

The first of these problems solution in the general case is hardly feasible, but in specific special cases it can be obtained.

Consider, for example, the simplest problem of optimal route choosing out of two possible ones. In this case, possible solutions set  $Q$  contains two elements:  $q_1$  – first route is selected,  $q_2$  – second route is selected. Let us set a one-coordinate phase space whose points represent to cost of transporting a unit of cargo from a supplier to a consumer. Let the random values of transportation costs for first and second routes be evenly distributed:

$$f_1(C_1) = \begin{cases} \frac{1}{2a_1}, & C_1 \in [C_1^{(0)} - a_1, C_1^{(0)} + a_1], \\ 0, & C_1 \notin [C_1^{(0)} - a_1, C_1^{(0)} + a_1]; \end{cases} \quad (7)$$

$$f_2(C_2) = \begin{cases} \frac{1}{2a_2}, & C_2 \in [C_2^{(0)} - a_2, C_2^{(0)} + a_2], \\ 0, & C_2 \notin [C_2^{(0)} - a_2, C_2^{(0)} + a_2]. \end{cases} \quad (8)$$

Suppose that  $C_1^{(0)} < C_2^{(0)}$  and therefore first route is chosen as optimal one. Let us estimate the stability of this solution.

Possible options for graphical display of the task are shown in Fig. 2. The shaded areas in Fig. 2 contain sets of points  $(C_1, C_2)$ , for which  $C_1 < C_2$ . Probabilities of getting into these areas determine the consideration degree of choice  $q_1$ . These probabilities are calculated by the formula

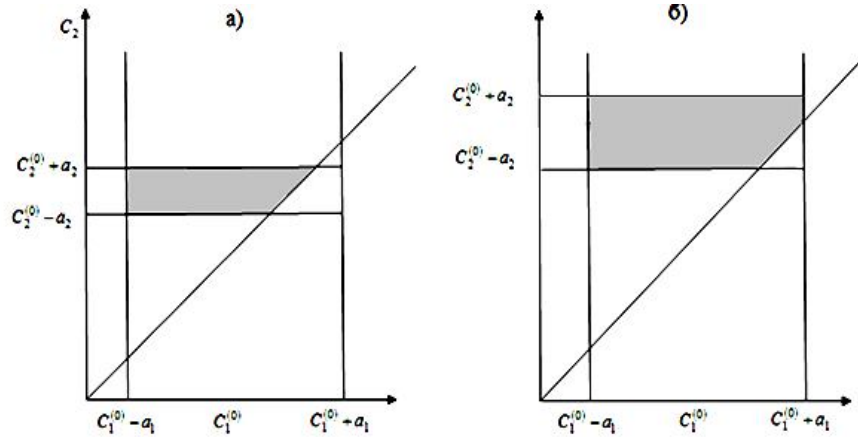


Fig. 2. Graphical displays of the route selection problem

$$P(C_1 < C_2) = \int_{C_1^{(0)} - a_1}^{C_2^{(0)} - a_2} \frac{1}{2a_1} dC_1 + \int_{C_2^{(0)} - a_2}^{\min(C_2^{(0)} + a_2, C_1^{(0)} + a_1)} \frac{1}{2a_1} dC_1 \int_{C_1}^{C_2^{(0)} + a_2} \frac{1}{2a_2} dC_2 = \frac{1}{2a_1} (C_2^{(0)} - a_2 - C_1^{(0)} + a_1) + \int_{C_2^{(0)} - a_2}^{\min(C_2^{(0)} + a_2, C_1^{(0)} + a_1)} \frac{1}{4a_1 a_2} (C_2^{(0)} + a_2 - C_1) dC_1 = (9) = \frac{1}{2a_1} (C_2^{(0)} - a_2 - C_1^{(0)} + a_1) + \left[ \frac{1}{4a_1 a_2} (C_2^{(0)} + a_2) \times \left[ \left( \min(C_2^{(0)} + a_2, C_1^{(0)} + a_1) \right) - (C_2^{(0)} - a_2) \right] - \frac{1}{2} \left[ \left( \min(C_2^{(0)} + a_2, C_1^{(0)} + a_1) \right)^2 - (C_2^{(0)} - a_2)^2 \right] \right].$$

The analysis of the obtained expression shows that its maximum value equal to 1 (corresponds to the absolute solution stability) is achieved when (Fig. 3, a):

$$C_2^{(0)} - a_2 = C_1^{(0)} + a_1, \text{ that is } C_2^{(0)} - C_1^{(0)} = a_1 + a_2.$$

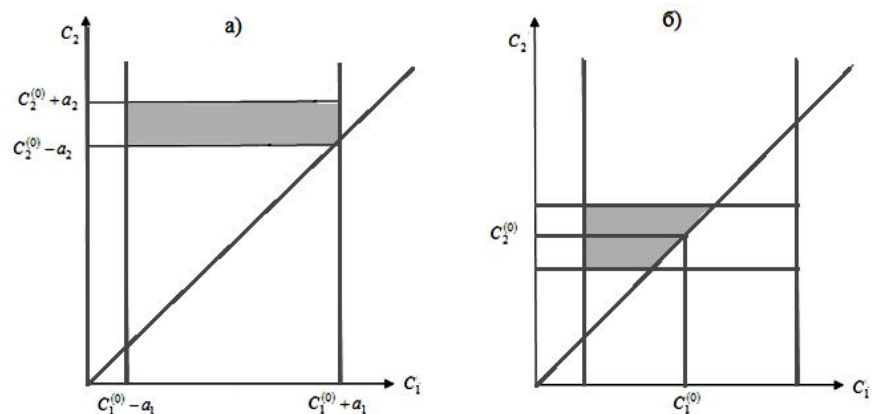


Fig. 3. Graphical display of options for route selection tasks

Wherein  $P(C_1 \leq C_2) = P(C_1 > C_2)$ .

The proposed method for solutions stability calculating of control problems, adopted under conditions of uncertainty in initial data, can be used when choosing for optimal option from a discrete set of possible ones. Let us determine, for example, stability of made decision in problem of optimal route choosing considered above. Let us simplify it as much as possible by reducing it to a two-alternative one. Let there are two competing routes for which random cost of transportation distribution densities are given (7), (8) with specific values

$$C_1^{(0)} = 11, a_1 = 5; C_2^{(0)} = 12, a_2 = 2.$$

Let the first route be chosen as optimal one. Let us determine solution stability. Specifying the general relation (9) for this case, we have

$$\begin{aligned} P(C_1 < C_2) &= \int_{C_2^{(0)}-a_2}^{C_2^{(0)}+a_2} \frac{1}{2a_1} dC_2 \int_{C_1^{(0)}-a_1}^{C_2} \frac{1}{2a_1} dC_1 = \\ &= \frac{1}{4a_1a_2} \int_{C_2^{(0)}-a_2}^{C_2^{(0)}+a_2} (C_2 - C_1^{(0)} + a_1) dC_2 = \quad (10) \\ &= \left( 2a_2C_2^{(0)} - (C_1^{(0)} - a_1)2a_2 \right) / (4a_1a_2) = \\ &= \left( C_2^{(0)} - C_1^{(0)} + a_1 \right) / 2a_1 = 1/2 + \left( C_2^{(0)} - C_1^{(0)} \right) / 2a_1. \end{aligned}$$

Wherein  $P(C_1 < C_2) = 1/2 + 1/10 = 0.6$ .

The obtained value of stability in this particular case is not large, due to small difference between  $C_1^{(0)}$  and  $C_2^{(0)}$  and large value of  $a_1$  parameter. The proposed method for assessing solution stability of the problem can be generalized in case when the cost requirement for the selected route is formed more stringently, particularly:  $kC_1 < C_2$ ,  $k > 1$ . In this case, the calculation formula (10) for assessing stability will take the form:

$$\begin{aligned} P(kC_1 < C_2) &= \int_{C_2^{(0)}-a_2}^{C_2^{(0)}+a_2} \frac{1}{2a_2} dC_2 \int_{C_1^{(0)}-a_1}^{C_2/k} \frac{1}{2a_1} dC_1 = \\ &= \frac{1}{4a_1a_2} \int_{C_2^{(0)}-a_2}^{C_2^{(0)}+a_2} \left( \frac{C_2}{k} - C_1^{(0)} + a_1 \right) dC_2 = \\ &= \left( 2a_2C_2^{(0)}/k - (C_1^{(0)} - a_1)2a_2 \right) / (4a_1a_2) = \\ &= \frac{C_2^{(0)}/k - C_1^{(0)} + a_1}{2a_1} = \frac{1}{2} + \frac{C_2^{(0)}/k - C_1^{(0)}}{2a_1}. \end{aligned}$$

Moreover, for the same initial data ( $C_1^{(0)} = 11$ ,  $C_2^{(0)} = 12$ ,  $a_1 = 5$ ) and  $k = 1,09$  the value of stability level, the solution will be equal to  $P(kC_1 < C_2) = \frac{1}{2} + \frac{12/1.09 - 11}{10} \approx \frac{1}{2}$  that is, the choice of first route is unstable.

Directions for further research may be related to consideration of problems in cases where similar ones are inaccurate in the sense of Pavlak [16=19].

A possible approach to solving the problems arising in this case was proposed in [20, 21].

## Conclusions

Proposed method for finding optimal routes when the initial data are deterministic or random variables with known distribution densities.

A particular case of a probabilistic-theoretical description of the initial data is considered, when a simple solution to the problem can be obtained.

The given method provides an approximate solution in the general case for arbitrary distribution densities of random initial data.

Considered situation when initial data are not clearly defined. A simple computational procedure for obtaining a solution is gained.

A method for assessing the solutions stability to control problems adopted under conditions of uncertainty in initial data is considered.

## REFERENCES

1. Yudin, D.B. and Golshtein, E.G. (1969), *Issue of linear programming transport type*, Nauka, Moscow, 384 p.
2. Sira, O.V. (2010), *Multidimensional models of logistics in conditions of uncertainty*, FOP Stetsenko I.I., Kharkiv, 512 p.
3. Raskin, L.G. and Kirichenko I.O. (1982), "Multi-index problems of linear programming", *Radio and communication*, Moscow, 240 p.
4. Raskin, L.G. (1976), *Analysis of complex systems and control theory elements*, Sov. radio, Moscow, 344 p.
5. Raskin, L.G. and Sira O.V. (2008), *Fuzzy mathematics*, Parus, Kharkiv, 352 p.
6. Zadeh, L.A. (1965), "Fuzzy sets", *Information and Control*, Vol. 8, pp. 338-353.
7. Dubois, D. and Prade, A. (1990), "Theory of possibilities. Application to the representation of knowledge in computer science", *Radio and communication*, Moscow, 286 p.
8. Kofman, A. (1982), "Introduction to the theory of fuzzy sets", *Radio and communication*, Moscow, 486 p.
9. Liu, B. and Zhao, R. (1995), *Stochastic Programming and Fuzzy Programming*, Tsinghua University Press, 312 p.
10. Raskin, L.G. (1988), *Mathematical methods for the study of operations and analysis of complex air defense weapons systems*, VIRTА, Kharkiv, 177 p.
11. Kachanov, N.A. (2000), *Optimal control of the state of dynamic systems under uncertainty*, Techno-Art, Kharkiv, 209 p.
12. Kushner, G.J. (1969), *Stochastic stability and control*, Mir, Moscow, 201p.
13. Zubarev, V.V., Kovtunenکو, A.P. and Raskin, L.G. (2005), *Mathematical methods for assessing and predicting technical indicators of the operational properties of radio engineering systems*, NAU, Kyiv, 184 p.
14. Bodyansky, E.V. and Boryachok, M.D. (1993), *Optimal control of stochastic objects under uncertainty*, ISDO, Kyiv, 162 p.
15. Letov A.M. (1981), *Mathematical theory of control process*, Nauka, Moscow, 256 p.

16. Pawlak, Z. (1982), *Rough sets: International Journal of Information and Computer Sciences*, Vol.11, No.5, pp. 341–356.
17. Pawlak, Z. (1991), *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publisher, Dordrecht, 284 p
18. Slowinski, R. and Vanderpooten, D. (2000), “A generalized definition of rough approximations based on similarity”, *IEEE Transactions on Knowledge and data Engineering*, Vol. 12, No. 2, pp. 331–336, DOI: <https://doi.org/10.1109/69.842271>.
19. Alefeld, G. and Herzberger, J. (1983), *Introduction to Interval Computations*, Academic Press, New York, 352 p.
20. Raskin, Lev and Sira Oksana (2016), “Fuzzy models of rough mathematics”, *Eastern-European Journal of Enterprise Technologies*, Vol. 6, Issue 4, pp. 53-60, DOI: <https://doi.org/10.15587/1729-4061.2016.86739>.

Received (Надійшла) 18.11.2020

Accepted for publication (Прийнята до друку) 03.02.2021

#### ВІДОМОСТІ ПРО АВТОРІВ / ABOUT THE AUTHORS

- Раскін Лев Григорович** – доктор технічних наук, професор, професор кафедри розподілених інформаційних систем і хмарних технологій, Національний технічний університет «Харківський політехнічний інститут», Харків, Україна;  
**Lev Raskin** – Doctor of Technical Sciences, Professor, Professor of Distributed Information Systems and Cloud Technologies Department, National Technical University «Kharkiv Polytechnic Institute», Kharkiv, Ukraine;  
e-mail: [topologyr@ukr.net](mailto:topologyr@ukr.net); ORCID ID: <http://orcid.org/0000-0002-9015-4016>.
- Сіра Оксана Володимирівна** – доктор технічних наук, професор, професор кафедри розподілених інформаційних систем і хмарних технологій, Національний технічний університет «Харківський політехнічний інститут», Харків, Україна;  
**Oksana Sira** – Doctor of Technical Sciences, Professor, Professor of Distributed Information Systems and Cloud Technologies Department, National Technical University «Kharkiv Polytechnic Institute», Kharkiv, Ukraine;  
e-mail: [topologys@ukr.net](mailto:topologys@ukr.net); ORCID ID: <http://orcid.org/0000-0002-4869-2371>.
- Парфенюк Юрій Леонідович** – аспірант кафедри розподілених інформаційних систем і хмарних технологій, Національний технічний університет «Харківський політехнічний інститут», Харків, Україна;  
**Yurii Parfeniuk** – Postgraduate Student of Distributed Information Systems and Cloud Technologies Department, National Technical University «Kharkiv Polytechnic Institute», Kharkiv, Ukraine;  
e-mail: [parfuriy.l@gmail.com](mailto:parfuriy.l@gmail.com); ORCID ID: <http://orcid.org/0000-0001-5357-1868>.

#### Вибір оптимального маршруту у розподіленій транспортній мережі в умовах невизначеності

Л. Г. Раскін, О. В. Сіра, Ю. Л. Парфенюк

**Анотація. Актуальність.** Для заданого набору значень довжин ділянок розгалуженої транспортної мережі розроблений точний метод відшукування оптимальних маршрутів. Метод забезпечує отримання наближеного рішення, коли вихідні дані - випадкові величини з відомими законами розподілу, а також, якщо ці дані задані нечітко. Для окремого випадку з нормальним розподілом числових характеристик мережі рішення доведено до кінцевих результатів. **Метод.** Запропоновано точний метод детермінованої маршрутизації, що дає наближене рішення, якщо вихідні дані випадкові. Метод поширений на випадок, коли вихідні дані описані в термінах теорії нечітких множин. Розглянуто проблему оцінки стійкості рішень задач теорії управління в умовах невизначеності вихідних даних. **Результати.** Запропоновано метод відшукування оптимальних маршрутів, коли вихідні дані - детерміновані або випадкові величини з відомими щільностями розподілу. Розглянуто окремий випадок теоретико-імовірнісного опису вихідних даних, коли може бути отримано просте рішення задачі. Запропоновано метод отримання наближеного рішення в загальному випадку для довільних щільності розподілу випадкових вихідних даних. Розглянуто ситуацію, коли вихідні дані визначені нечітко. Запропоновано проста обчислювальна процедура отримання рішення. Розглянуто метод оцінки стійкості рішень задач управління, прийнятих в умовах невизначеності вихідних даних.

**Ключові слова:** транспортна мережа; оптимальний маршрут; похідні дані – випадкові або нечіткі числа; стійкість рішень задач управління.

#### Выбор оптимального маршрута в разветвленной транспортной сети в условиях неопределенности

Л. Г. Раскин, О. В. Сера, Ю. Л. Парфенюк

**Аннотация. Актуальность.** Для заданного набора значений длин участков разветвленной транспортной сети разработан точный метод отыскания оптимальных маршрутов. Метод обеспечивает получение приближенного решения, когда исходные данные – случайные величины с известными законами распределения, а также, если эти данные заданы нечетко. Для частного случая с нормальным распределением числовых характеристик сети решение доведено до конечных результатов. **Метод.** Предложен точный метод детерминированной маршрутизации, дающий приближенное решение, если исходные данные случайны. Метод распространен на случай, когда исходные данные описаны в терминах теории нечетких множеств. Рассмотрена проблема оценки устойчивости решений задач теории управления в условиях неопределенности исходных данных. **Результаты.** Предложен метод отыскания оптимальных маршрутов, когда исходные данные – детерминированные или случайные величины с известными плотностями распределения. Рассмотрен частный случай теоретико-вероятностного описания исходных данных, когда может быть получено простое решение задачи. Предложен метод получения приближенного решения в общем случае для произвольных плотностей распределения случайных исходных данных. Рассмотрена ситуация, когда исходные данные определены нечетко. Предложена простая вычислительная процедура получения решения. Рассмотрен метод оценки устойчивости решений задач управления, принятых в условиях неопределенности исходных данных.

**Ключевые слова:** транспортная сеть; оптимальный маршрут; исходные данные – случайные или нечеткие числа; устойчивость решений задач управления.